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A NEW CLASS OF CONTACT RIEMANNIAN MANIFOLDS

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ABSTRACT

N. Tanaka ([10]) defined the canonical affine connection on a nondegenerate integrable CR manifold. In the present paper, we introduce a new class of contact Riemannian manifolds satisfying (C) $(\hat{\nabla}_{\dot\gamma}R)(\cdot,\dot\gamma)\dot\gamma = 0$ for any unit $\hat{\nabla}$-geodesic $\gamma(\hat{\nabla}_{\dot\gamma}\dot\gamma = 0)$, where $\hat{\nabla}$ is the generalized Tanaka connection. In particular, when the associated CR structure of a given contact Riemannian manifold is integrable we have a structure theorem and find examples which are neither Sasakian nor locally symmetric but satisfy the condition (C).

1. Introduction

A Riemannian manifold $M = (M, g)$ with Riemannian metric tenor g is called (E. Cartan [6]) a locally symmetric space if M satisfies $\nabla R = 0$, where ∇ is the Levi-Civita connection. In [8] it was proved that a Sasakian manifold (or normal contact Riemannian manifold) which is locally symmetric must have constant curvature 1. This fact means that local symmetry is a very strong condition for a Sasakian manifold. For this reason, T. Takahashi ([9]) introduced the notion of Sasakian locally ϕ-symmetric spaces which may be considered as the analogues of locally Hermitian symmetric spaces. A contact Riemannian locally ϕ-symmetric space is defined as a generalization of the notion of the Sasakian locally ϕ-symmetric spaces and investigated in [4].

On the other hand, N. Tanaka ([10]) defined the canonical affine connection on a nondegenerate integrable CR manifold. In [12] S.Tanno defined the generalized Tanaka connection $\hat{\nabla}$ on a contact Riemannian manifold and further,

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he proved that for a given contact Riemannian manifold M the associated CR structure is strongly pseudo-convex integrable if and only if M satisfies the integrability condition $Q = 0$ (see section 2), in which case the connection $\hat{\nabla}$ coincides with the Tanaka connection. Here, we note that the normality of a contact Riemannian structure implies the integrability of the associated CR structure, but the converse does not always hold. The associated CR structres of 3-dimensional contact Riemannian manifolds are always integrable (see [12]). Also, we see that their associated CR structures are integrable for contact Riemannian manifolds with the characteristic vector field ξ belonging to the (k,μ)-nullity distribution (see [5]), that is, contact Riemannian manifolds which satisfy $R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$, where k,μ are constant and $2h$ is the Lie derivative of ϕ in the direction ξ. A Sasakian manifold, in particular, is determined by $k = 1$ $(h = 0)$.

Recently, in [1] a locally symmetric space M was characterized by the remarkable property that the Jacobi operator field $R_{\dot\gamma} = R(\cdot,\dot\gamma)\dot\gamma$ is diagonalizable by a ∇-parallel orthonormal frame field along γ and its eigenvalues are constant along γ for any geodesic γ on M. In the present paper, we introduce a new class of contact Riemannian manifolds satisfying

$$\text{(C)} \qquad (\hat{\nabla}_{\dot\gamma}R)(\cdot,\dot\gamma)\dot\gamma = 0$$

for any unit $\hat{\nabla}$-geodesic $\gamma(\hat{\nabla}_{\dot\gamma}\dot\gamma = 0)$, where $\hat{\nabla}$ is the generalized Tanaka connection. We observe that the geodesics of the Levi-Civita connection and the generalized Tanaka connection do not coincide in general, and further, that a contact Riemannian manifold satisfies the condition (C) for any $\hat{\nabla}$-geodesic γ if and only if the Jacobi operator field $R_{\dot\gamma}$ is diagonalizable by a $\hat{\nabla}$-parallel orthonormal frame field along γ and its eigenvalues are constant along γ for any $\hat{\nabla}$-geodesic γ in the manifold. In section 3 we prove that a Sasakian manifold M satisfies condition (C) for any $\hat{\nabla}$-geodesic γ if and only if M is locally ϕ-symmetric. In section 4 we prove

THEOREM A: *Let M be a contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution. If M satisfies the condition* (C) *for any $\hat{\nabla}$-geodesic γ, then one of the following holds :*

(i) $k = 1$ *and M is a Sasakian locally ϕ-symmetric space;*
(ii) $\mu = 0$ *and M is a 3-dimensional locally ϕ-symmetric space (in the sense of [4]);*
(iii) $\mu = 2$ *and M is a locally ϕ-symmetric space (in the sense of [4]).*

It is worth mentioning that a contact Riemannian manifold M^{2n+1} $(n \geq 2)$ satisfying $R(X,Y)\xi = 0$ for all vector fields X and Y(i.e., ξ belonging to the (0,0)-nullity distribution) is locally symmetric but does not satisfy the condition (C) for any $\hat{\nabla}$-geodesic γ (see Remark in section 4).

In [5] the authors showed that the characteristic vector field ξ of the tangent sphere bundle T_1M with the standard contact Riemannian structure belongs to the (k,μ)-nullity distribution if and only if the base manifold M is of constant curvature c, in which case $k = c(2-c)$ and $\mu = -2c$. Thus applying Theorem A to this result, we have

COROLLARY B: *Let M be a space of constant curvature c. If the standard contact Riemannian structure of the tangent sphere bundle T_1M satisfies the condition* (C) *for any $\hat{\nabla}$-geodesic γ, then one of the following holds :*

(i) *the base manifold M is 2-dimensional and T_1M is of constant curvature 1 $(c=1)$;*
(ii) *the base manifold M is 2-dimensional and T_1M is flat $(c=0)$;*
(iii) *the base manifold M is of constant curvature -1 and T_1M is locally ϕ-symmetric $(c=-1)$.*

In [13] it has been proved that the gauge invariant B of (1,3)-type of the standard contact Riemannian structure on T_1M vanishes if and only if the base manifold M is of constant curvature -1. Particularly in section 5, we prove

THEOREM C: *Let M be a 2-dimensional Riemannian manifold. If the standard contact Riemannian structure of the tangent sphere bundle T_1M satisfies the condition* (C) *for any $\hat{\nabla}$-geodesic γ, then M has constant Gauss curvature $\kappa = 1$, 0 or -1.*

Lastly in this work, we can show that contact Riemannian manifolds with ξ belonging to the $(k,2)$-nullity distribution $(k \neq 1)$ including the standard contact Riemannian structure of the tangent sphere bundle T_1M of M with constant curvature -1 are examples that are neither Sasakian nor locally symmetric but satisfy condition (C) for any $\hat{\nabla}$-geodesic γ. All manifolds in the present paper are assumed to be connected and of class C^∞.

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2. Preliminaries

A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, which is called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there exists an associated Riemannian metric g and a $(1,1)$-type tensor field ϕ such that

$$(2.1) \qquad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi,$$

where X and Y are vector fields on M. From (2.1) it follows that

$$(2.2) \qquad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (2.1) is said to be a contact Riemannian manifold and is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold M, we define a $(1,1)$-type tensor field h by $h = \frac{1}{2} L_\xi \phi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$(2.3) \qquad h\xi = 0 \quad \text{and} \quad h\phi = -\phi h,$$

$$(2.4) \qquad \nabla_X \xi = -\phi X - \phi h X,$$

where ∇ is the Levi-Civita connection. From (2.3) and (2.4) we see that each trajectory of ξ is a geodesic.

A contact Riemannian manifold for which ξ is Killing is called a K-contact Riemannian manifold. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For a contact Riemannian manifold M one may define naturally an almost complex structure J on $M \times \mathbb{R}$,

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be normal or Sasakian. It is known that M is normal if and only if M satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. A Sasakian manifold is characterized by a condition

$$(2.5) \qquad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

for all vector fields X and Y on the manifold. We denote by R the Riemannian curvature tensor of M defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z on M. It is well-known that M is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.6}$$

for all vector fields X and Y. For more details about contact Riemannian manifolds we refer to [2], [11], [12], etc.

For a contact Riemannian manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = \mathfrak{D}_p \oplus \{\xi\}_p$(direct sum), where we denote $\mathfrak{D}_p = \{v \in T_pM | \eta(v) = 0\}$. Then $\mathfrak{D} : p \to \mathfrak{D}_p$ defines a distribution orthogonal to ξ. The $2n$-dimensional distribution $\mathfrak{D}$ is called the **contact distribution**. We see that the restriction $\bar{\phi} = \phi|\mathfrak{D}$ of ϕ to $\mathfrak{D}$ defines an almost complex structure to $\mathfrak{D}$, and further see that the associated Levi form, which is defined by $L(X,Y) = -d\eta(X, \bar{\phi}Y)$, $X, Y \in \mathfrak{D}$, is positive definite and hermitian. We call the pair $(\eta, \bar{\phi})$ a **strongly pseudo-convex, pseudo-hermitian structure** on M. Since $d\eta(\phi X, \phi Y) = d\eta(X,Y)$, we see that $[\bar{\phi}X, \bar{\phi}Y] - [X,Y] \in \mathfrak{D}$ for $X, Y \in \mathfrak{D}$. Further, if M satisfies the condition

$$[\bar{\phi}, \bar{\phi}](X,Y) = 0$$

for $X, Y \in \mathfrak{D}$, then the pair $(\eta, \bar{\phi})$ is called the **strongly pseudo-convex integrable CR structure** (associated with the contact Rimannian structure (η, g)). Taking account of (2.5) we see that for a Sasakian manifold the associated CR structure is strongly pseudo-convex integrable. Now we review the generalized Tanaka connection ([12]) on a contact Riemannian manifold $M = (M; \eta, g)$. The **generalized Tanaka connection** $\hat{\nabla}$ is defined as

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X and Y on M. Together with (2.4), $\hat{\nabla}$ is rewritten by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)(\phi X + \phi h X) - g(\phi X + \phi h X, Y)\xi \tag{2.7}$$

and we see that the generalized Tanaka connection $\hat{\nabla}$ has torsion $\hat{T}(X,Y) = 2g(X, \phi Y)\xi + \eta(Y)\phi h X - \eta(X)\phi h Y$. We put

$$A(X,Y) = \eta(X)\phi Y + \eta(Y)(\phi X + \phi h X) - g(\phi X + \phi h X, Y)\xi \tag{2.8}$$

for all vector fields X and Y on M. Then A is a (1,2)-type tensor field on M and $\hat{\nabla}_X Y = \nabla_X Y + A(X,Y)$. In particular, for a K-contact Riemannian manifold we see that

$$A(X,Y) = \eta(X)\phi Y + \eta(Y)\phi X - g(\phi X, Y)\xi, \tag{2.9}$$

where X and Y are vector fields.

It was obtained (Proposition 2.1 in [12]) that for a given contact Riemannian manifold M the associated CR structure is strongly pseudo-convex integrable if and only if M satisfies the integrability condition $Q = 0$, where Q is a (1,2)-type tensor field on M defined by

$$Q(X,Y) = (\nabla_X \phi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX)$$

for all vector fields X, Y on M. Further, it was proved that

Proposition 2.1 ([12]): *The generalized Tanaka connection $\hat{\nabla}$ on a contact Riemannian manifold $M = (M; \eta, g)$ is the unique linear connection satisfying the following:*

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;

(ii) $\hat{\nabla} g = 0$;

(iii-1) $\hat{T}(X,Y) = 2d\eta(X,Y)\xi$, X, $Y \in \mathfrak{D}$;

(iii-2) $\hat{T}(\xi, \phi Y) = -\phi \hat{T}(\xi, Y)$, $Y \in \mathfrak{D}$;

(iv) $(\hat{\nabla}_X \phi)Y = Q(X,Y)$, X, $Y \in TM$.

The Tanaka connection ([10]) on a nondegenerate integrable CR manifold is defined as a unique linear connection satisfying (i), (ii), (iii-1), (iii-2) and $\hat{\nabla}\phi = 0$. So, $\hat{\nabla}$ is a naturally generalized one of the Tanaka connection. For more details about the generalized Tanaka connection we refer to [12].

Let γ be a $\hat{\nabla}$-geodesic parametrized with the arc-length parameter s, where a $\hat{\nabla}$-geodesic means a geodesic with respect to $\hat{\nabla}$. From (2.7) we see that a $\hat{\nabla}$-geodesic does not coincide with a ∇-geodesic in general. Define the Jacobi operator $R_{\dot{\gamma}}$ by $R_{\dot{\gamma}} = R(\cdot, \dot{\gamma})\dot{\gamma}$ along γ, where $\dot{\gamma}$ is the tangent vector field of γ. Then $R_{\dot{\gamma}}$ is a symmetric $(1,1)$-type tensor field along γ. Moreover, from (i) of Proposition 2.1 we observe that $\eta(\dot{\gamma})$ is constant along γ, and thus a $\hat{\nabla}$-geodesic whose tangent initially belongs to $\mathfrak{D}$ remains in $\mathfrak{D}$. We call such a $\hat{\nabla}$-geodesic which is tangent to $\mathfrak{D}$ a **horizontal $\hat{\nabla}$-geodesic**.

Here we recall the definition of a Sasakian locally ϕ-symmetric space ([9]).

Definition 2.2: A Sasakian manifold $M = (M;\eta, g)$ is said to be locally ϕ-symmetric if $\phi^2(\nabla_V R)(X,Y)Z = 0$ for all vector fields $V, X, Y, Z \in \mathfrak{D}$.

As a generalization of the above Sasakian one, a contact Riemannian locally ϕ-symmetric space is defined in [4] by the same curvature condition.

In [5] the (k,μ)-nullity distribution of a contact Riemannian manifold M, for the pair $(k,\mu) \in \mathbb{R}^2$, is defined by

$$N(k,\mu) : p \to N_p(k,\mu) = \{z \in T_pM | R(x,y)z = k(g(y,z)x - g(x,z)y) + \mu(g(y,z)hx - g(x,z)hy) \text{ for any } x, y \in T_pM\}.$$

A contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution satisfies

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY). \tag{2.10}$$

It is shown that ([5]) a contact Riemannian manifold satisfying (2.10) is obtained by applying a D-homothetic deformation ([11]) on a contact Riemannian manifold with $R(X,Y)\xi = 0$. It is well-known that the tangent sphere bundle of a flat Riemannian manifold admits a contact Riemannian structure satisfying $R(X,Y)\xi = 0$. In [5] it is also shown that the (k,μ)-nullity condition for ξ remains invariant under a D-homothetic deformation. Furthermore, in [5] they showed

THEOREM 2.3: *Let $M = (M;\eta, g)$ be a contact Riemannian manifold. If ξ belongs to the (k,μ)-nullity distribution, then $k \le 1$. If $k = 1$, then $h = 0$ and M is a Sasakian manifold. If $k < 1$, then M admits three mutually orthogonal and integrable distributions $\mathfrak{D}(0)$, $\mathfrak{D}(\lambda)$ and $\mathfrak{D}(-\lambda)$, defined by the eigenspaces of h, where $\lambda = \sqrt{1-k}$. Moreover,*

$$\begin{aligned}
&R(X_\lambda, Y_\lambda)Z_{-\lambda} = (k-\mu)\{g(\phi Y_\lambda, Z_{-\lambda})\phi X_\lambda - g(\phi X_\lambda, Z_{-\lambda})\phi Y_\lambda\},\\
&R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = (k-\mu)\{g(\phi Y_{-\lambda}, Z_\lambda)\phi X_{-\lambda} - g(\phi X_{-\lambda}, Z_\lambda)\phi Y_{-\lambda}\},\\
&R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = kg(\phi X_\lambda, Z_{-\lambda})\phi Y_{-\lambda} + \mu g(\phi X_\lambda, Y_{-\lambda})\phi Z_{-\lambda},\\
&R(X_\lambda, Y_{-\lambda})Z_\lambda = -kg(\phi Y_{-\lambda}, Z_\lambda)\phi X_\lambda - \mu g(\phi Y_{-\lambda}, X_\lambda)\phi Z_\lambda,\\
&R(X_\lambda, Y_\lambda)Z_\lambda = \{2(1+\lambda) - \mu\}\{g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda\},\\
&R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = \{2(1-\lambda) - \mu\}\{g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}\},
\end{aligned}$$

where X_λ, Y_λ, $Z_\lambda \in \mathfrak{D}(\lambda)$ and $X_{-\lambda}$, $Y_{-\lambda}$, $Z_{-\lambda} \in \mathfrak{D}(-\lambda)$.

For more details about a contact Riemannian manifold satisfying (2.10), we refer to [5].

3. A Sasakian manifold satisfying the condition (C)

In this section, we prove

THEOREM 3.1: *Let M be a Sasakian manifold. Then M is locally ϕ-symmetric if and only if M satisfies the condition* (C) *for any horizontal $\hat{\nabla}$-geodesic γ, or if and only if M satisfies the condition* (C) *for any $\hat{\nabla}$-geodesic γ.*

Proof: Let M be a Sasakian manifold and let γ be a horizontal $\hat{\nabla}$-geodesic parametrized by the arc-length parameter s. Then from (2.7) and (2.9) we see that $A(V,V)=0$ for any vector field $V\in\mathfrak{D}$, and thus we see that a horizontal $\hat{\nabla}$-geodesic coincides with a ∇-geodesic. From (2.9) we have

(3.1)
$$\begin{aligned}(\hat{\nabla}_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma =&\hat{\nabla}_{\dot\gamma}R(V,\dot\gamma)\dot\gamma - R(\hat{\nabla}_{\dot\gamma}V,\dot\gamma)\dot\gamma\\ =&\nabla_{\dot\gamma}R(V,\dot\gamma)\dot\gamma + A(\dot\gamma,R(V,\dot\gamma)\dot\gamma) - R(\nabla_{\dot\gamma}V,\dot\gamma)\dot\gamma - R(A(\dot\gamma,V),\dot\gamma)\dot\gamma\\ =&(\nabla_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma - g(\phi\dot\gamma,R(V,\dot\gamma)\dot\gamma)\xi + g(\phi\dot\gamma,V)R(\xi,\dot\gamma)\dot\gamma\end{aligned}$$

for any $V\in\mathfrak{D}$.

Now let us assume that M satisfies the condition (C), $(\hat{\nabla}_{\dot\gamma}R)(\cdot,\dot\gamma)\dot\gamma=0$, for any horizontal $\hat{\nabla}$-geodesic. Thus from (3.1), taking account of (2.6), we have $g((\nabla_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma,W)=0$ for any $V,W\in\mathfrak{D}$, which implies $g((\nabla_u R)(v,u)u,w)=0$ for any $u,v,w\in\mathfrak{D}_p$ and any $p\in M$. And hence $g(\nabla_U R)(V,U)U,W)=0$ for any $U,V,W\in\mathfrak{D}$. By the polarization and first and second Bianchi identities, we have

$$g((\nabla_U R)(V,X)Y,W)=0$$

for any $U,V,X,Y,W\in\mathfrak{D}$ (cf. [1], [15]). Thus we see that M is locally ϕ-symmetric.

Conversely, let us assume that M is a locally ϕ-symmetric space. Then from (2.6) and (3.1) we have

$$g((\hat{\nabla}_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma,W)=0 \tag{3.2}$$

for any $V,W\in\mathfrak{D}$.

From (3.1) we get

$$\begin{aligned}g((\hat{\nabla}_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma,\xi)=g((\nabla_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma,\xi)-g(\phi\dot\gamma,R(V,\dot\gamma)\dot\gamma)\\ +g(\phi\dot\gamma,V)g(R(\xi,\dot\gamma)\dot\gamma,\xi)\end{aligned} \tag{3.3}$$

for any $V\in\mathfrak{D}$. From (3.3) together with (2.6) we have

$$g((\hat{\nabla}_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma,\xi)=g((\nabla_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma,\xi)-g(\phi\dot\gamma,R(V,\dot\gamma)\dot\gamma)+g(\phi\dot\gamma,V). \tag{3.4}$$

On the other hand, from (2.6), taking account of (2.4), we have

$$(\nabla_{\dot\gamma}R)(V,\dot\gamma)\xi = -R(\nabla_{\dot\gamma}V,\dot\gamma)\xi + R(V,\dot\gamma)\phi\dot\gamma \tag{3.5}$$

and

$$R(\nabla_{\dot\gamma}V,\dot\gamma)\xi = -g(V,\phi\dot\gamma)\dot\gamma \tag{3.6}$$

for any $V \in \mathfrak{D}$. Since $g((\nabla_{\dot\gamma}R)(V,\dot\gamma)\xi,\dot\gamma) = -g((\nabla_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma,\xi)$, from (3.4), (3.5) and (3.6), we have

$$g((\hat\nabla_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma,\xi) = 0. \tag{3.7}$$

Also, from (2.7), (2.9) and (i) of Proposition 2.1 we have

$$\begin{aligned}(\hat\nabla_{\dot\gamma}R)(\xi,\dot\gamma)\dot\gamma &= \nabla_{\dot\gamma}R(\xi,\dot\gamma)\dot\gamma + A(\dot\gamma,R(\xi,\dot\gamma)\dot\gamma)\\ &= \nabla_{\dot\gamma}\xi + A(\dot\gamma,\xi).\end{aligned} \tag{3.8}$$

Since $g(A(\dot\gamma,\xi),\xi) = 0$, from (3.8), together with (2.3), we get

$$g((\hat\nabla_{\dot\gamma}R)(\xi,\dot\gamma)\dot\gamma,\xi) = 0. \tag{3.9}$$

Therefore from (3.2), (3.7) and (3.9), we conclude that

$$(\hat\nabla_{\dot\gamma}R)(\cdot,\dot\gamma)\dot\gamma = 0$$

for any horizontal $\hat\nabla$-geodesic γ.

Further, from (2.6) and (i) of Proposition 2.1 we see that

$$(\hat\nabla_{\xi}R)(Y,X)\xi = 0$$

for all vector fields X and Y on M. Then it suffices to prove $g((\hat\nabla_\xi R)(Y,V)V,X) = 0$ for all vector fields $V,X,Y \in \mathfrak{D}$. It follows from (2.7) and (2.9) that

(3.10)

$$\begin{aligned}g((\hat\nabla_\xi R)(Y,V)V,X) =&(\nabla_\xi R)(Y,V)V,X) + g(\phi R(Y,V)V,X) - g(R(\phi Y,V)V,X)\\ &- g(R(Y,\phi V)V,X) - g(R(Y,V)\phi V,X)\end{aligned}$$

for all vector fields $V,X,Y \in \mathfrak{D}$. From (2.4), (2.6) and the second Bianchi identity, we have

(3.11)

$$\begin{aligned}((\nabla_\xi R)(Y,V)V,X) =&g(\phi Y,V)g(V,X) - g(\phi Y,X)g(V,V) + g(R(V,X)\phi Y,V)\\ &+ g(\phi V,X)g(V,Y) - g(R(V,X)\phi V,Y).\end{aligned}$$

Thus, from (3.10) and (3.11), we have

(3.12)
$$((\hat{\nabla}_\xi R)(Y,V)V,X) = g(\phi Y,V)g(V,X) - g(\phi Y,X)g(V,V) + g(\phi V,X)g(V,Y) - g(R(Y,V)\phi V,X) + g(\phi R(Y,V)V,X)$$

for all vector fields $V, X, Y \in \mathfrak{D}$. From the definition of the curvature tensor, taking account of (2.4) and (2.5), we obtain

(3.13)
$$R(Y,X)\phi Z - \phi R(Y,X)Z = g(\phi Y,Z)X - g(X,Z)\phi Y - g(\phi X,Z)Y + g(Y,Z)\phi X,$$

where X, Y and Z are vector fields on M. By using (3.13), from (3.12) we see that $g((\hat{\nabla}_\xi R)(Y,V)V,X) = 0$ for all vector fields $V, X, Y \in \mathfrak{D}$. ∎

4. A contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution

In the present section we prove Theorem A. The following Lemma was proved in [5].

LEMMA 4.1: *Let $M = (M^{2n+1}, \phi, \xi, \eta, g)$ be a contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution. Then for any vector fields X, Y we have*

$$(4.1) \qquad (\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \text{ (or } Q(X,Y) = 0),$$

$$(4.2) \qquad (\nabla_X h)Y = \{(1-k)g(X,\phi Y) + g(X,h\phi Y)\}\xi + \eta(Y)h(\phi X + \phi hX) - \mu\eta(X)\phi hY.$$

At first, we generalize Theorem 3.1. Namely, we prove

PROPOSITION 4.2: *Let M be a contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution. Then M satisfies the condition (C) for any horizontal $\hat{\nabla}$-geodesic if and only if M is locally ϕ-symmetric (in the sense of [4]).*

Proof: Let M be a contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution, i.e.,

$$(4.3) \qquad R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

for any vector fields X, Y on M, where $(k,\mu) \in \mathbb{R}^2$. Let γ be a $\hat{\nabla}$-geodesic parametrized by arc-length parameter s with $\dot{\gamma}(0) \in \mathfrak{D}_{\gamma(0)}$. Then we see that $\dot{\gamma}(s) \in \mathfrak{D}_{\gamma(s)}$ and, also from (2.7), we have

$$\nabla_{\dot{\gamma}}\dot{\gamma} = g(\phi h\dot{\gamma},\dot{\gamma})\xi. \tag{4.4}$$

From (2.7) and (4.4) we have

(4.5)
$$\begin{aligned}(\hat{\nabla}_{\dot{\gamma}}R)(V,\dot{\gamma})\dot{\gamma} =& \hat{\nabla}_{\dot{\gamma}}R(V,\dot{\gamma})\dot{\gamma} - R(\hat{\nabla}_{\dot{\gamma}}V,\dot{\gamma})\dot{\gamma}\\ =& \nabla_{\dot{\gamma}}R(V,\dot{\gamma})\dot{\gamma} + A(\dot{\gamma},R(V,\dot{\gamma})\dot{\gamma}) - R(\nabla_{\dot{\gamma}}V,\dot{\gamma})\dot{\gamma} - R(A(\dot{\gamma},V),\dot{\gamma})\dot{\gamma}\\ =& (\nabla_{\dot{\gamma}}R)(V,\dot{\gamma})\dot{\gamma} + g(\phi h\dot{\gamma},\dot{\gamma})R(V,\xi)\dot{\gamma} + g(\phi h\dot{\gamma},\dot{\gamma})R(V,\dot{\gamma})\xi\\ & - g(\phi\dot{\gamma} + \phi h\dot{\gamma}, R(V,\dot{\gamma})\dot{\gamma})\xi + g(\phi\dot{\gamma}+\phi h\dot{\gamma},V)R(\xi,\dot{\gamma})\dot{\gamma}\end{aligned}$$

for any $V \in \mathfrak{D}$.

Now let us assume that M satisfies the condition (C), $(\hat{\nabla}_{\dot{\gamma}}R)(\cdot,\dot{\gamma})\dot{\gamma} = 0$, for any horizontal $\hat{\nabla}$-geodesic. Thus from (4.5), taking account of (4.3), we have $g((\nabla_{\dot{\gamma}}R)(V,\dot{\gamma})\dot{\gamma},W) = 0$ for any $V,W \in \mathfrak{D}$, which implies $g((\nabla_u R)(v,u)u,w) = 0$ for any $u,v,w \in \mathfrak{D}_p$ and any $p \in M$. And hence $g(\nabla_U R)(V,U)U,W) = 0$ for any $U,V,W \in \mathfrak{D}$. Similarly as in the proof of Theorem 3.1, we have

$$g((\nabla_U R)(V,X)Y,W) = 0$$

for any $U,V,X,Y,W \in \mathfrak{D}$. Thus we see that M is locally ϕ-symmetric (in the sense of [4]).

Conversely, let us assume that M is a locally ϕ-symmetric space (in the sense of [4]). Then from (4.3) and (4.5) we have

$$g((\hat{\nabla}_{\dot{\gamma}}R)(V,\dot{\gamma})\dot{\gamma},W) = 0 \tag{4.6}$$

for any $V,W \in \mathfrak{D}$.

Also from (4.5), we get

(4.7)
$$\begin{aligned}g((\hat{\nabla}_{\dot{\gamma}}R)(V,\dot{\gamma})\dot{\gamma},\xi) =& g((\nabla_{\dot{\gamma}}R)(V,\dot{\gamma})\dot{\gamma},\xi) + g(\phi h\dot{\gamma},\dot{\gamma})g(R(V,\xi)\dot{\gamma},\xi)\\ & - g(\phi\dot{\gamma}+\phi h\dot{\gamma},R(V,\dot{\gamma})\dot{\gamma}) + g(\phi\dot{\gamma}+\phi h\dot{\gamma},V)g(R(\xi,\dot{\gamma})\dot{\gamma},\xi)\end{aligned}$$

for any $V \in \mathfrak{D}$. From (4.7) together with (4.3) we have

(4.8)
$$\begin{aligned}g((\hat{\nabla}_{\dot{\gamma}}R)(V,\dot{\gamma})\dot{\gamma},\xi) =& g((\nabla_{\dot{\gamma}}R)(V,\dot{\gamma})\dot{\gamma},\xi) - kg(\phi h\dot{\gamma},\dot{\gamma})g(V,\dot{\gamma})\\ & - \mu g(\phi h\dot{\gamma},\dot{\gamma})g(hV,\dot{\gamma})\\ & - g(\phi\dot{\gamma}+\phi h\dot{\gamma},R(V,\dot{\gamma})\dot{\gamma}) + kg(\phi\dot{\gamma}+\phi h\dot{\gamma},V)\\ & + \mu g(\phi\dot{\gamma}+\phi h\dot{\gamma},V)g(h\dot{\gamma},\dot{\gamma}).\end{aligned}$$

On the other hand, from (4.3), taking account of (2.4) and (4.4), we have

$$(4.9)\quad (\nabla_{\dot\gamma}R)(V,\dot\gamma)\xi = -R(\nabla_{\dot\gamma}V,\dot\gamma)\xi - g(\phi h\dot\gamma,\dot\gamma)R(V,\xi)\xi + R(V,\dot\gamma)(\phi\dot\gamma+\phi h\dot\gamma)$$

and

(4.10)

$$\begin{aligned}R(\nabla_{\dot\gamma}V,\dot\gamma)\xi &= -k\{\eta(\nabla_{\dot\gamma}V)\dot\gamma\} - \mu\{\eta(\nabla_{\dot\gamma}V)h\dot\gamma\}\\ &= -k\{g(V,\phi\dot\gamma+\phi h\dot\gamma)\dot\gamma\} - \mu\{g(V,\phi\dot\gamma+\phi h\dot\gamma)h\dot\gamma\}\end{aligned}$$

for any $V \in \mathfrak{D}$. Since $g((\nabla_{\dot\gamma}R)(V,\dot\gamma)\xi,\dot\gamma) = -g((\nabla_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma,\xi)$, from (4.8), (4.9) and (4.10) we have

$$(4.11)\qquad g((\hat\nabla_{\dot\gamma}R)(V,\dot\gamma)\dot\gamma,\xi) = 0.$$

Also, from (2.7), (i) of Proposition 2.1 and (4.3) we have

(4.12)

$$\begin{aligned}(\hat\nabla_{\dot\gamma}R)(\xi,\dot\gamma)\dot\gamma &= \nabla_{\dot\gamma}R(\xi,\dot\gamma)\dot\gamma + A(\dot\gamma,R(\xi,\dot\gamma)\dot\gamma)\\ &= \nabla_{\dot\gamma}(k\xi+\mu g(h\dot\gamma,\dot\gamma)\xi) + kA(\dot\gamma,\xi) + \mu g(h\dot\gamma,\dot\gamma)A(\dot\gamma,\xi)\\ &= k\nabla_{\dot\gamma}\xi + \mu\{g((\nabla_{\dot\gamma}h)\dot\gamma,\dot\gamma)\xi + 2g(h\nabla_{\dot\gamma}\dot\gamma,\dot\gamma)\xi + g(h\dot\gamma,\dot\gamma)\nabla_{\dot\gamma}\xi\}\\ &\quad + kA(\dot\gamma,\xi) + \mu g(h\dot\gamma,\dot\gamma)A(\dot\gamma,\xi).\end{aligned}$$

Since $g(A(\dot\gamma,\xi),\xi) = 0$, from (4.12), together with (2.3), (4.2) and (4.4), we get

$$(4.13)\qquad g((\hat\nabla_{\dot\gamma}R)(\xi,\dot\gamma)\dot\gamma,\xi) = 0.$$

Therefore from (4.6), (4.11) and (4.13), we conclude that

$$(\hat\nabla_{\dot\gamma}R)(\cdot,\dot\gamma)\dot\gamma = 0$$

for any $\hat\nabla$-geodesic γ with $\dot\gamma \in \mathfrak{D}_{\gamma(s)}$. ∎

Now we prove Theorem A. From (2.3), (2.7) and (4.2) we get

$$(4.14)\qquad \begin{aligned}(\hat\nabla_\xi h)X &= \hat\nabla_\xi hX - h\hat\nabla_\xi X\\ &= (\nabla_\xi h)X + A(\xi,hX) - hA(\xi,X)\\ &= (\nabla_\xi h)X + (\phi h - h\phi)X\\ &= (2-\mu)\phi hX\end{aligned}$$

for any vector field X on M. From (2.3), (4.3) and (4.14) we see that

$$(\hat{\nabla}_\xi R)(X,\xi)\xi = \mu(2-\mu)\phi hX \tag{4.15}$$

for any vector field X on M.

We suppose that M satisfies the condition (C) for any $\hat{\nabla}$-geodesic. Then from (4.15) it follows that

$$\mu(2-\mu)\phi hX = 0$$

for any vector field X on M.

If $\mu = 0$, then from (4.3) we see that ξ belongs to the k (or $(k,0)$)-nullity distribution. Namely,

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) \tag{4.16}$$

for any vector fields X, Y on M, where k is a real number. From (i) of Proposition 2.1 and (4.16) we easily see that

$$(\hat{\nabla}_\xi R)(X,Y)\xi = 0$$

for all vector fields X and Y on M. Thus, by virtue of Proposition 4.2, it only remains to examine $g((\hat{\nabla}_\xi R)(X,V)V,Y) = 0$ for all vector fields $V,X,Y \in \mathfrak{D}$. It follows from (2.7), together with (2.2), that

$$\begin{aligned} g((\hat{\nabla}_\xi R)(X,V)V,Y) =&(\nabla_\xi R)(X,V)V,Y) + g(\phi R(X,V)V,Y) \\ &- g(R(\phi X,V)V,Y) - g(R(X,\phi V)V,Y) \\ &- g(R(X,V)\phi V,Y) \end{aligned} \tag{4.17}$$

for all vector fields $V,X,Y \in \mathfrak{D}$. On the other hand, from (4.16) and the second Bianchi identity we obtain

$$\begin{aligned} g((\nabla_\xi R)(X,V)V,Y) =&k\{g(\phi X + \phi hX,V)g(V,Y) - g(\phi X + \phi hX,Y)g(V,V) \\ &- g(\phi hV,V)g(Y,X) + g(\phi V + \phi hV,Y)g(V,X)\} \\ &+ g(R(V,Y)\phi Y,X) + g(R(V,Y)\phi hY,X) \\ &- g(R(V,Y)\phi V,X) - g(R(V,Y)\phi hV,X), \end{aligned} \tag{4.18}$$

where $X,Y \in \mathfrak{D}$. From the definition of the curvature tensor, taking account of (2.4) and (4.1), we obtain

$$\begin{aligned} &g(R(X,Y)\phi Z,W) - g(\phi R(X,Y)Z,W) \\ &\quad=g(\phi X + \phi hX,Z)g(Y+hY,W) - g(Y+hY,Z)g(\phi X + \phi hX,W) \\ &\quad\quad- g(\phi Y + \phi hY,Z)g(X+hX,W) + g(X+hX,Z)g(\phi Y + \phi hY,W), \end{aligned} \tag{4.19}$$

where $X, Y, Z, W \in \mathfrak{D}$. From (4.17), (4.18) and (4.19), we have

$$\begin{aligned} & g((\hat{\nabla}_\xi R)(X,V)V,Y) \\ & = (k-1)\{g(\phi X + \phi hX, V)g(V,Y) - g(\phi X + \phi hX, Y)g(V,V) \\ & \quad - g(\phi hV, V)g(Y,X) + g(\phi V + \phi hV, Y)g(V,X)\} \\ & \quad - g(\phi X + \phi hX, V)g(hV,Y) + g(\phi X + \phi hX, Y)g(hV,V) \\ & \quad + g(\phi hV, V)g(hY,X) - g(\phi V + \phi hV, Y)g(hV,X) \\ & \quad - g(R(V,Y)\phi hV, X) + g(R(V,Y)\phi hX, V), \end{aligned} \tag{4.20}$$

for all vector fields $V, X, Y \in \mathfrak{D}$. Here we divide our arguments into two cases: (i) dim $M = 2n+1 = 3$, (ii) dim $M = 2n+1 \geq 5$.

Case (i): We assume that $hX = \lambda X$ and $V = \phi X$ ($\|X\| = 1$). Then from (2.3) and (4.20) we can see that $g((\hat{\nabla}_\xi R)(X,V)V,Y) = 0$.

Case (ii): If $k = 1$, then from (2.6) we see that M is Sasakian. So, now we suppose that $k \neq 1$ and $g((\hat{\nabla}_\xi R)(X,V)V,Y) = 0$. Taking account of Theorem 2.3 and $2n+1 \geq 5$, we assume that $hX = \lambda X$ and $hV = \lambda V$, where Y and V are unit and mutually orthogonal. Then from (4.20) we obtain

$$\begin{aligned} & (k-1)(1+\lambda)g(X,\phi Y) \\ & = \lambda(1+\lambda)g(X,\phi Y) + \lambda g(\phi R(V,Y)X, V) - \lambda g(R(V,Y)\phi X, V). \end{aligned} \tag{4.21}$$

Also, from (4.19) we have

$$g(\phi R(V,Y)X, V) - g(R(V,Y)\phi X, V) = (1-\lambda^2)g(X,\phi Y). \tag{4.22}$$

The equations (4.21) and (4.22), together with $\lambda = \sqrt{1-k}$, yield

$$\lambda(\lambda+1) = 0,$$

and hence $\lambda = 0$. Together with (2.3) we see that $h = 0$ and $k = 1$, which is a contradiction. Thus M is Sasakian and further, by virtue of Theorem 3.1, locally ϕ-symmetric.

Next we consider the case $\mu = 2$. Let M be a contact Riemannian manifold with ξ belonging to the $(k,2)$-nullity distribution, i.e.,

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + 2(\eta(Y)hX - \eta(X)hY), \tag{4.23}$$

where k is a real number. Then from (4.14) we find that $\hat{\nabla}_\xi h = 0$, and hence from (4.23) we see that

$$(\hat{\nabla}_\xi R)(X,Y)\xi = 0$$

for all vector fields X and Y on M. Thus, also by virtue of Proposition 4.2, it only remains to examine $g((\hat{\nabla}_\xi R)(X,V)V,Y) = 0$ for all vector fields $V, X, Y \in \mathfrak{D}$. Similarly as in the case $\mu = 0$, we have

(4.24)
$$\begin{aligned} g((\nabla_\xi R)(X,V)V,Y) =& k\{g(\phi X + \phi hX, V)g(V,Y) - g(\phi X + \phi hX, Y)g(V,V) \\ & - g(\phi hV, V)g(Y,X) + g(\phi V + \phi hV, Y)g(V,X)\} \\ & + 2\{g(\phi X + \phi hX, V)g(hV,Y) \\ & - g(\phi X + \phi hX, Y)g(hV,V) \\ & - g(\phi hV, V)g(hY,X) + g(\phi V + \phi hV, Y)g(hV,X)\} \\ & + g(R(V,Y)\phi Y, X) + g(R(V,Y)\phi hY, X) \\ & - g(R(V,Y)\phi V, X) - g(R(V,Y)\phi hV, X) \end{aligned}$$

where $X, Y \in \mathfrak{D}$. And further, we have

$$\begin{aligned} & g((\hat{\nabla}_\xi R)(X,V)V,Y) \\ & = (k-1)\{g(\phi X + \phi hX, V)g(V,Y) - g(\phi X + \phi hX, Y)g(V,V) \\ & \quad - g(\phi hV, V)g(Y,X) + g(\phi V + \phi hV, Y)g(V,X)\} \\ & \quad + g(\phi X + \phi hX, V)g(hV,Y) - g(\phi X + \phi hX, Y)g(hV,V) \\ & \quad - g(\phi hV, V)g(hY,X) + g(\phi V + \phi hV, Y)g(hV,X) \\ & \quad - g(R(V,Y)\phi hV, X) + g(R(V,Y)\phi hX, V), \end{aligned} \tag{4.25}$$

for all vector fields $V, X, Y \in \mathfrak{D}$. By linearization of (4.25), we have

$$\begin{aligned} & (\hat{\nabla}_\xi R)(X,V)W + (\hat{\nabla}_\xi R)(X,W)V \\ & = (k-1)\{g(\phi X + \phi hX, V)W + g(\phi X + \phi hX, W)V \\ & \quad - 2g(V,W)(\phi X + \phi hX) - 2g(\phi hV, W)X \\ & \quad + g(V,X)(\phi W + \phi hW) + g(W,X)(\phi V + \phi hV)\} \\ & \quad + g(\phi X + \phi hX, V)g(hW,Y) + g(\phi X + \phi hX, W)g(hV,Y) \\ & \quad - 2g(hV,W)(\phi X + \phi hX) - 2g(\phi hV, W)hX \\ & \quad + g(hV,X)(\phi W + \phi hW) + g(hW,X)(\phi V + \phi hV) \\ & \quad - R(\phi hV, X)W - R(\phi hW, X)V \\ & \quad - R(\phi hX, V)W - R(\phi hX, W)V, \end{aligned} \tag{4.26}$$

for all vector fields $X, V, W \in \mathfrak{D}$. By using Theorem 2.3 in the last two lines in (4.26) we find that $(\hat{\nabla}_\xi R)(X,V)W + (\hat{\nabla}_\xi R)(X,W)V = 0$ in all six possible cases. Thus we see that a contact Riemannian manifold with ξ belonging to the $(k, 2)$-nullity distribution always satisfies the condition (C) for any $\hat{\nabla}$-geodesic γ.

Finally, if $\phi hX = 0$, then we easily see that $h = 0$ and, from (4.1), M is a Sasakian manifold. Further, from Theorem 3.1, M is locally ϕ-symmetric. ∎

Remark: A contact flat Riemannian structure, for example, is explicitly expressed as $\mathbb{R}^3(x^1, x^2, x^3)$ with $\eta = \frac{1}{2}(\cos x^3 dx^1 + \sin x^3 dx^2)$ and $g_{ij} = \frac{1}{4}\delta_{ij}$. For dimension ≥ 5, a contact manifold cannot admit a contact Riemannian structure of vanishing curvature (cf. [2]). Also, it was proved that a contact Riemannian manifold M^{2n+1} which satisfies $R(X,Y)\xi = 0$ for all vector fields X and Y (i.e., ξ belonging to the (0,0)-nullity distribution) is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant sectional curvature equal to 4. We see that a contact Riemannian manifold M^{2n+1} $(n \geq 2)$ satisfying $R(X,Y)\xi = 0$ is locally symmetric but does not satisfy the condition (C) for any $\hat{\nabla}$-geodesic.

It is well-known that the tangent sphere bundle T_1M of a Riemannian manifold M has a contact Riemannian structure in a natural way (cf. Chapter VII of [2]). We recall a result of Y. Tashiro ([14]) that the standard contact Riemannian structure on T_1M is K-contact if and only if M has constant curvature 1, in which case the structure on T_1M is Sasakian. In [5] we find that the characteristic vector field ξ of the tangent sphere bundle T_1M with the standard contact Riemannian structure belongs to the (k, μ)-nullity distribution if and only if M is of constant curvature c, in which case $k = c(2-c)$ and $\mu = -2c$. Also, it was proved in [3] that the standard contact Riemannian structure of T_1M is locally symmetric if and only if either the base manifold is flat or the base manifold is 2-dimensional and of constant curvature 1. Thus, by applying Theorem A to the tangent sphere bundle T_1M of a Riemannian manifold M with constant curvature we have Corollary B.

5. The tangent sphere bundle of a 2-dimensional Riemannian manifold

In this section we prove Theorem C. First we examine explicitly the tangent sphere bundle of a 2-dimensional Riemannian manifold. Let M be a 2-dimensional Riemannian manifold, TM be the tangent bundle with an almost Kähler structure $(J, \bar{g})$ (cf. [7]) and T_1M be the tangent sphere bundle of M (i.e., the set of all unit tangent vectors of M) with the projection map $\pi : T_1M \to M$. Let (x^1, x^2) be an isothermal local coordinate system on M such that the

Riemannian metric is of the form

$$f^2((dx^1)^2 + (dx^2)^2),$$

where f is a positive-valued function on M. Then, by a straightfoward calculation, we see that the Gauss curvature κ of M is $-(\Delta_0 \log f)/f^2$ where Δ_0 is the Laplacian with respect to the Euclidean metric. For any point $z \in T_1M$ in TM, we let (u^1, u^2, y^1, y^2) be a local coordinate system around z such that $u^i = x^i \circ \pi$ and (y^1, y^2) is the fiber coordinate with $f^2((y^1)^2 + (y^2)^2) = 1$. The vector field

$$N = y^1 \frac{\partial}{\partial y^1} + y^2 \frac{\partial}{\partial y^2}$$

is a unit normal and position vector for the point z of T_1M. Denote by g' the metric of T_1M induced from $\bar{g}$ (Sasaki metric) on TM. Define ϕ', ξ', η' by

$$JN = -\xi', \quad JX = \phi' X + \eta'(X) N \tag{5.1}$$

for any vector field X on T_1M. Then we see $g'(X, \phi' Y) = 2d\eta'(X, Y)$. By a simple rectification, namely, $\eta = \frac{1}{2}\eta'$, $\xi = 2\xi'$, $\phi' = \phi$ and $g = \frac{1}{4}g'$, we have a contact Riemannian tangent sphere bundle $T_1M = (T_1M, \eta, g)$. Also, taking account of (5.1) and the definitions of J and g, we have a local orthonormal frame field $\{e_1, e_2, e_3\}$ such that

$$\begin{aligned} e_3 = \xi &= 2\sum_{ijk} \left(y^i \frac{\partial}{\partial u^i} - \widetilde{\{ijk\}} y^j y^k \frac{\partial}{\partial y^i} \right), \\ e_1 &= 2\sum_i v^i \frac{\partial}{\partial y^i}, \\ e_2 = \phi e_1 &= -2\sum_{ijk} \left(v^i \frac{\partial}{\partial u^i} - \widetilde{\{ijk\}} y^j v^k \frac{\partial}{\partial y^i} \right), \end{aligned} \tag{5.2}$$

for $i, j, k = 1, 2$ where $(v^1, v^2) = (-y^2, y^1)$, $\widetilde{\{ijk\}} = \{ijk\} \circ \pi$ and $\{ijk\}$ are Christoffel symbols of Riemannian connection of M. For the local orthonormal frame field, we have

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2\tilde{\kappa} e_1, \quad [e_3, e_1] = 2e_2 \tag{5.3}$$

where $\tilde{\kappa} = \kappa \circ \pi$. Put

$$\Gamma_{ijk} = g(\nabla_{e_i} e_j, e_k) \quad \text{for } i, j, k = 1, 2, 3.$$

Then we have $\Gamma_{ijk} = -\Gamma_{ikj}$. We recall the formula

$$\begin{aligned}2g(\nabla_X Y, Z) =& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)\\ &+ g(Y,[Z,X]) + g(Z,[X,Y]) - g(X,[Y,Z])\end{aligned}$$

for all vector fields X, Y, Z on T_1M. Using this formula, we have

$$\Gamma_{123} = 2 - \tilde{\kappa}, \quad \Gamma_{213} = \Gamma_{321} = -\tilde{\kappa}, \quad \text{all other } \Gamma_{ijk} \text{ being zero.} \tag{5.4}$$

Moreover, from (5.3) and (5.4) we have

$$\begin{aligned} R(e_1, e_3)e_3 &= \tilde{\kappa}^2 e_1 - (e_3\tilde{\kappa})e_2,\\ R(e_2, e_3)e_3 &= -(e_3\tilde{\kappa})e_1 - (3\tilde{\kappa}^2 - 4\tilde{\kappa})e_2,\\ R(e_2, e_1)e_1 &= \tilde{\kappa}^2 e_2,\\ R(e_3, e_1)e_1 &= \tilde{\kappa}^2 e_3,\\ R(e_1, e_2)e_2 &= \tilde{\kappa}^2 e_1 + (e_2\tilde{\kappa})e_3,\\ R(e_3, e_2)e_2 &= (e_2\tilde{\kappa})e_1 - (3\tilde{\kappa}^2 - 4\tilde{\kappa})e_3,\\ R(e_1, e_2)e_3 &= -(e_2\tilde{\kappa})e_2,\\ R(e_3, e_1)e_2 &= -(e_3\tilde{\kappa})e_3,\\ R(e_2, e_3)e_1 &= (e_2\tilde{\kappa})e_2 + (e_3\tilde{\kappa})e_3. \end{aligned} \tag{5.5}$$

From (i) of Proposition 2.1 and (5.4), we have

$$\hat{\nabla}\xi = 0 \quad \text{and} \quad \hat{\nabla}_{e_i} e_j = 0 \quad \text{for } i, j = 1, 2. \tag{5.6}$$

Thus we see that e_1, e_2, e_3 are all $\hat{\nabla}$-geodesic vector fields. We prove

PROPOSITION 5.1: *The tangent sphere bundle T_1M of a 2-dimensional Riemannian manifold M satisfies* (C) *for any horizontal $\hat{\nabla}$-geodesic γ if and only if the Gauss curvature κ of M is constant.*

Proof: Suppose that T_1M satisfies (C) for any horizontal $\hat{\nabla}$-geodesic γ. Then from (5.5) and (5.6) we easily see that κ is constant. Conversely, if the Gauss curvature κ is constant, then also from (5.5) and (5.6) we see that T_1M satisfies $(\hat{\nabla}_{e_i} R)(\cdot, e_j)e_k = 0$ for $i, j, k = 1, 2$, i.e., $(\hat{\nabla}_v R)(\cdot, v)v = 0$ for any $v \in \mathfrak{D}_p$ and $p \in M$. Thus we have

$$(\hat{\nabla}_{\dot{\gamma}} R)(\cdot, \dot{\gamma})\dot{\gamma} = 0$$

for any horizontal $\hat{\nabla}$-geodesic γ. ∎

Now we prove Theorem C. From (2.7) and (5.4) we get

$$\hat{\nabla}_\xi e_1 = \nabla_\xi e_1 + e_2 = (\tilde{\kappa}+1)e_2, \quad \hat{\nabla}_\xi e_2 = \nabla_\xi e_2 - e_1 = -(\tilde{\kappa}+1)e_1. \tag{5.7}$$

Suppose that T_1M satisfies (C) for any $\hat{\nabla}$-geodesic γ. Then from Proposition 5.1 we first see that κ is constant. And, together with (5.5) and (5.7), we calculate

$$(\hat{\nabla}_\xi R)(e_i, \xi)\xi = 4\tilde{\kappa}(\tilde{\kappa}-1)(\tilde{\kappa}+1)e_j,$$

where $i, j = 1, 2$ and $i \neq j$. Thus we see that $\kappa = 0$, 1 or -1. Conversely, if $\kappa = 0$ or 1, then from (5.5) it follows that T_1M is flat or of constant curvature 1, respectively. Or if $\kappa = -1$, then from (5.7) we see that $\hat{\nabla}_\xi e_i = 0$, $i = 1, 2$. And from (5.5) and (5.6) we easily see $\hat{\nabla}R = 0$. Therefore we have proved Theorem C.

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